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Scale-covariant field theories: I. Overview

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Abstract. We examine some of the problems with scale-covariant quantum field theories, as developed by Klauder, and summarise our progress in solving them.

1. Introduction

This is the first of a series of papers (Ebbutt and Rivers 1982a, b, c, d) (henceforth known as II, III, IV, V respectively) in which we examine some aspects of scale-covariant quantum field theory, as developed by Klauder in several pioneering papers over the last decade. (For a review of Klauder's work see Klauder (1979a, b) and references therein.)

Scale-covariant quantisation is intended to give an understanding of the problems of ultraviolet non-renormalisable field theories. Independent of the computational scheme to be adopted, it is argued that, in the path integral formulation of such theories, certain paths that are available to the field in the free theory become inadmissible. That is, the functional measure has to be changed, the translation-invariant measures of the renormalisable canonical theory being replaced by *scale-covariant* measures.

This choice of scale-covariant measures is motivated by the operator-product expansion (Klauder 1979b, 1981a) and leads to affine commutation relations (as distinct from normal ordering of the operator products which leads to canonical commutation relations).

We thus wish to calculate formal path integrals of the kind (taking the Euclidean theory as an example)

$$Z[h]=\int \mathscr{D}'[\phi]\exp-\left(A[\phi]-\mathrm{i}\int h\phi\right) \tag{1.1}$$

where A is a classical action, and $\mathscr{D}'[\phi]$ the scale-covariant measure satisfying

$$\mathscr{D}'[\Lambda\phi]=F[\Lambda]\mathscr{D}'[\phi] \qquad \text{for} \quad \Lambda(x)>0, \forall x. \tag{1.2}$$

Empirically, we know that only Gaussian measures can be integrated. Since $\mathscr{D}'[\phi]$ can be formally expressed (non-uniquely) in terms of the translation-invariant measure $\mathscr{D}[\phi]$, satisfying

$$\mathscr{D}[\Lambda+\phi]=\mathscr{D}[\phi] \tag{1.3}$$

as

$$\mathscr{D}'[\phi]=\frac{\mathscr{D}[\phi]}{\Pi_x|\phi(x)|^\beta} \qquad 0<\beta\leqslant 1 \tag{1.4}$$

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we see that there is nothing overtly Gaussian about (1.1). Not surprisingly, scale-covariant integrals like (1.1) have proved markedly resistant to solution.

We can break the problem down into two parts. Firstly, we can try to solve (1.1) for a quadratic $A[\phi]$:

$$A_0[\phi] = \int dx\,(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2). \tag{1.5}$$

This describes the *pseudo-free* theory, most conveniently thought of as the $\lambda \to 0$ limit of a theory with ϕ self-interaction coupling strength λ †. To be specific, we shall only consider the $\lambda\phi^4$ interaction with action

$$A[\phi] = \int dx\,(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4). \tag{1.6}$$

Secondly, the assumption is made that we use the pseudo-free theory as a basis for developing a solution for the interacting theory. In particular, we might wish to establish a perturbation series in λ for the interacting theory about the pseudo-free theory.

In this paper we shall examine some of the problems with the pseudo-free and interacting scale-covariant theories with actions (1.5) and (1.6) and provide a summary of our progress in understanding them. The reader is referred to our work (Ebbutt and Rivers 1982a, b, c, d) for the detailed arguments. In general, our approach does not overlap very much with the existing literature on scale-covariant theories, although we are in continual debt to the work of Klauder (1979a, b, 1981a, b).

We have been primarily concerned with the following interrelated problems.

(i) Does the notion of discontinuous perturbations make sense in principle? That is, can we develop the perturbation series in λ for the scale-covariant theory with action (1.6) about the pseudo-free theory with action (1.5)? It has been argued by Nouri-Moghadam and Yoshimura (1978a) that Klauder's equations are so degenerate that it is not possible to develop such a series from them alone.

(ii) Can we develop some general understanding from particular cases? For example, Klauder (1977) has studied the effect of having a scale-*invariant* measure (i.e. $\beta = 1$ in (1.4)) in some detail). For this particular choice of β we see that, by introducing the auxiliary field χ, the scale-*invariant* measure is re-expressed in terms of translation-invariant measures $\mathscr{D}[\phi]$, $\mathscr{D}[\chi]$ as

$$\mathscr{D}'[\phi] = \int \mathscr{D}[\phi]\mathscr{D}[\chi] \exp -\tfrac{1}{2}\int \eta\phi^2\chi^2\,dx. \tag{1.7}$$

We expect that insofar as (1.7) can be interpreted as an expression of a 'hard-core' interaction the augmented formalism should permit greater physical insight. As we shall see, the case $\beta = 1$ arises naturally in some approximation schemes.

(iii) Does the change of measure affect the stability of the theory? This question is motivated by the observation that an alternative way to display the 'hard-core' is

† It was the notion of discontinuous perturbations (that switching off the interaction term in a non-renormalisable theory did not give the corresponding free theory) that was one of the basic ideas in developing scale-covariant theories. From this viewpoint, the failure of orthodox perturbation theory to control the ultraviolet infinities of non-renormalisable theories arises because we are expanding about the incorrect theory.

to write $\mathscr{D}'[\phi]$ in terms of translation-invariant measures as

$$\mathscr{D}'[\phi]=\mathscr{D}[\phi]\exp-\frac{\beta}{2}\int\delta(0)\ln\phi^2\,dx. \tag{1.8}$$

This has the interpretation of changing the 'classical' potential (for the translation-covariant theory) from $V(\phi)$ to (formally)

$$V'(\phi)=V(\phi)+\tfrac{1}{2}\beta\delta(0)\ln\phi^2. \tag{1.9}$$

As $\phi\to 0$, $V'(\phi)$ becomes unbounded below, suggesting instability.

(iv) So far, we have been mainly concerned with the pseudo-free theory. We shall see, in examining (ii) above, that the most convenient way to order contributions does not lend itself naturally to a λ-perturbation expansion. Rather, we have something more akin to mean-field or $1/N$ expansions†. This suggests that a more fruitful way of tackling the interacting scale-covariant theory will be via $1/N$-type expansions, rather than pseudo-perturbation theory. Nonetheless, these expansions are not wholly unrelated to the pseudo-free theory. The large-N limit of the generalised $O(N)$ theory is the last problem that we shall approach in this sequence of papers.

We conclude with the observation that all the above is not merely a pedagogic exercise. In quantum gravity we still have a theory in which renormalisation is poorly understood. Indeed, it has been argued that scale-covariant quantisation is particularly appropriate to gravity (Klauder 1970). A recent review article by Isham (1982) summarises these arguments succinctly. A solvable model using these ideas has been developed by Pilati (1981, 1982).

There is yet a further point. Grand unification has given rise to very large mass scales. It is arguable that, whatever the nature of the ultimate theory, at energies very much below these mass scales all theories are effectively renormalisable. It has been stated (without proof) (Nouri-Moghadam and Yoshimura 1979) that scale-invariant theories for which a mass scale M can arise naturally can show this behaviour. That is, there may be a large-M limit in which the scale-covariant theory can be identified with a translation-covariant theory. This would act to blur the distinction between renormalisable and non-renormalisable theories, and terminate a preoccupation that has dogged quantum field theory since its creation.

2. The scale-covariant equations and their pseudo-perturbation expansion

Consider the scale-covariant Euclidean theory of a single scalar field ϕ with generating functional

$$Z'[h]=\int\mathscr{D}'[\phi]\exp-\int dx\left(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\lambda_0\phi^4-ih\phi\right) \tag{2.1}$$

where $\mathscr{D}'[\phi]$ is normalised so that $Z'[0]=1$.

As always, functional integrals are deceptive in their implications for the unconnected Green functions

$$G_n(x_1\ldots x_n)=\frac{1}{i^n}\left.\frac{\delta^n Z'[h]}{\delta h(x_1)\ldots\delta h(x_n)}\right|_{h=0} \tag{2.2}$$

† In this way a mismatch between the incompatible definitions of operator products implied by scale-covariant and translation-covariant formalisms is avoided.

and the connected Green functions

$$W_n(x_1 \ldots x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n \ln Z'[h]}{\delta h(x_1) \ldots \delta h(x_n)}\bigg|_{h=0} \tag{2.3}$$

of the theory. Any exactly solvable models assume great importance in that they can give insight on the more realistic theory (2.1).

The only exactly solvable Euclidean-covariant model (for non-zero m_0, λ_0) is the independent-value model (IVM) in which the kinetic term is discarded, with generating functional

$$Z_0'[h] = \int \mathscr{D}'[\phi] \exp - \int \mathrm{d}x[\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 - \mathrm{i}h\phi]. \tag{2.4}$$

Since a Fock-space solution exists, it is known how to interpret the path integral, and it is found that $\mathscr{D}'[\phi]$ is scale-*invariant* (i.e. $\beta = 1$) for this important case.

Firstly, it is found that the equation satisfied by $Z_0'[h]$ has the form

$$\left\{h(x)\frac{\delta}{\delta h(x)} + :\frac{\delta}{\delta h(x)} m_0^2 \frac{\delta}{\delta h(x)}: - 4\lambda_0 :\frac{\delta^4}{\delta h(x)^4}:\right\} Z_0'[h] = 0 \tag{2.5}$$

where : : denotes the *subtraction* procedure

$$:\frac{\delta^p}{\delta h(x)^p}: Z'[h] = \frac{\delta^p Z'}{\delta h(x)^p} - \frac{\delta^p Z'}{\delta h(x)^p}\bigg|_{h=0} Z'. \tag{2.6}$$

Secondly, the *linear* formal equations for the *unconnected* Green functions $G_n(x_1 \ldots x_n)$ that follow from (2.5) are also satisfied by the *connected* Green functions $W_n(x_1 \ldots x_n)$. This is because, *for the* IVM, the W become progressively more multiplicatively singular as more points are made coincident, as $(x_i \neq x_j \neq x)$

$$W_{n+p}(xxxx \ldots xx_1 \ldots x_p) = \delta^{n-1}(0)\bar{W}_{n+p}(xxx \ldots xx_1 \ldots x_p) \tag{2.7}$$

where $\bar{W}$ is finite.

Reverting to the original theory (2.1) we assume similar properties.

Adopting the same subtraction procedure gives rise to the equations

$$\left\{h(x)\frac{\delta}{\delta h(x)} + :\frac{\delta}{\delta h(x)} K_x \frac{\delta}{\delta h(x)}: - 4\lambda_0 :\frac{\delta^4}{\delta h(x)^4}:\right\} Z_0'[h] = 0 \tag{2.8}$$

with $K_x = -\nabla_x^2 + m_0^2$.

Further assuming that the W_n get multiplicatively more singular, the greater the number of coincident points, equation (2.8) will give rise to linear equations for the W_n. These are

$$\left(\sum_{r=1}^{2m} \delta(x - x_r)\right) W_{2m}(x_1 \ldots x_{2m}) - \lim_{x' \to x} K_x W_{2m+2}(x'xx_1 \ldots x_{2m})$$
$$- 4\lambda_0 W_{2m+4}(xxxxx_1 \ldots x_{2m}) = 0 \qquad m \geqslant 1 \tag{2.9}$$

where we assume $W_{2m+1} = 0$.

They are represented diagrammatically in figure 1.

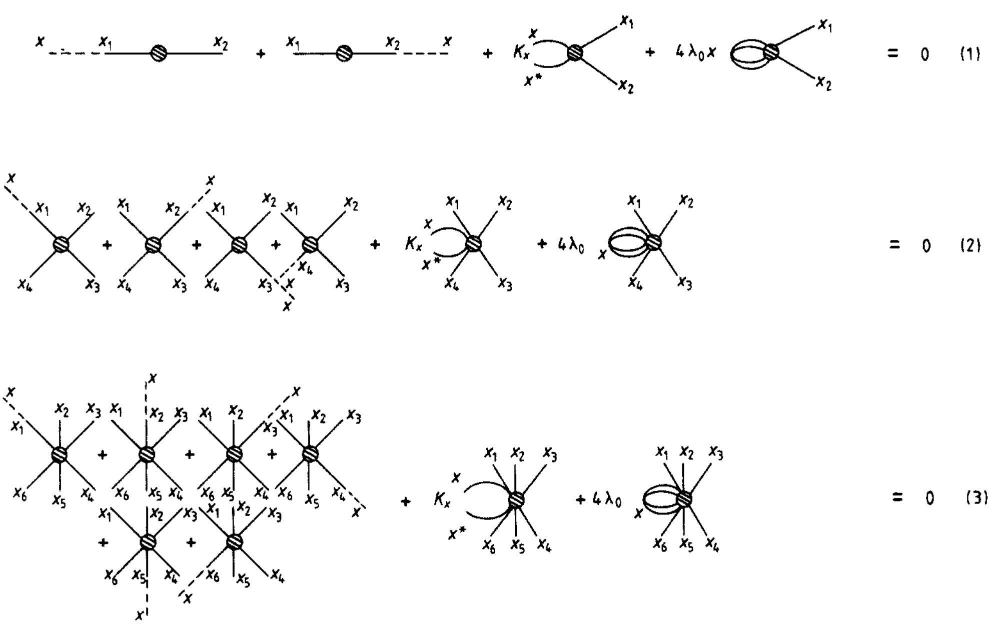

Figure 1. The diagrammatic description of the scale-covariant equations (2.9). Full lines denote ϕ fields. Circles denote connected Green functions. Broken lines denote δ functions.

These equations were examined in some detail by Nouri-Moghadam and Yoshimura (1978a). They concluded that they were so degenerate as to prohibit the unique development of a λ-perturbation series about the pseudo-free theory. That is, if

$$(W_{2n})_\lambda = \sum_p W_{2n}^{(p)} \lambda^p \qquad (2.10)$$

denotes the asymptotic perturbation series in λ, the $W_{2n}^{(p+1)}$ cannot be obtained uniquely, via (2.9), from the $W_{2n}^{(p)}$.

If correct this would have dramatic consequences, provided equation (2.8) expressed the total content of the functional integral (2.1). In paper II of this series we have re-examined this problem in some detail. A summary of our conclusion is presented below.

The first point to make is that requiring the perturbation series to exist order by order (and ignoring the problem of the resummation of the $(W_{2n})_\lambda$) is equivalent to imposing boundary conditions on (2.8). These serve to reduce the degeneracy of the equations for the $W_{2n}^{(p)}$†. Although the situation is not as bad as presented in (Nouri-Moghadam and Yoshimura 1978a), the conclusion that the $W_{2n}^{(p)}$ are *not* uniquely determined is correct.

We therefore need to supplement equations (2.9) in some way. One way is to impose the analogue of the renormalisation group and Callan–Symanzik equations. To see how this could work, we found it instructive to contrast the IVM of (2.4) with the translation-covariant static ultra-local model (SULM) of Caianiello and Scarpetta

† For example, in the Schwinger–Dyson equations for the canonical translation-covariant theory, W_2 is completely *undetermined*, but $(W_2)_\lambda$ is *uniquely* determined.

(1974a, b), with regularised generating functional

$$Z[h]=\int \mathscr{D}[\phi]\exp-\int \mathrm{d}x\left(\tfrac{1}{2}m_0^2\phi^2+\lambda_0\phi^4-\mathrm{i}h\phi\right). \tag{2.11}$$

Our conclusions are presented in tables 1 and 2. We see that, whereas the renormalisation-group-like (RG) equations contain no further information for the translation-covariant perturbation series, they provide much more for the scale-covariant perturbation series. The end result is that the scale-covariant perturbation series is determined up to an overall scale parameter, as we know from direct calculation.

We expect similar conclusions to be applicable to the more realistic scale-covariant theory of (2.1). That is, in principle the path integral (2.1) contains more information than is given in (2.9) and this can be expressed in renormalisation-group-type equations. However, we do not understand renormalisation well enough for scale-covariant theory to utilise this information.

There is an alternative approach to the problem of supplementing equations (2.9), with essentially the same information content†, for the particular case of the scale-invariant measure with $\beta=1$. Despite the results of the IVM (for which $\beta=1$) we

Table 1. The perturbation series for the ULM.

General Solution: W_2 completely undetermined	→ λ-perturbation series	All (W_n) uniquely determined (not uniquely summable)
↓ RG equations		↓ RG equations
W_2 determined up to a single parameter	→ λ-perturbation series	All (W_n) uniquely determined (uniquely summable)

Table 2. The perturbation series for the IVM.

General Solution: W_2, W_4 completely undetermined	→ λ-perturbation series	(W_2) completely undetermined. $(W_n)\cdot(W_2)^{-1}$ uniquely determined (not uniquely summable)
↓ RG equations		↓ RG equations
W_2, W_4 determined up to two parameters	→ λ-perturbation series	All (W_n) determined up to an arbitrary scale parameter ($(W_n)\cdot(W_2)^{-1}$ uniquely summable)

† In the limited sense of which W_n are unconstrained.

might consider dropping the subtraction procedure for this case, to give rise to the additional dynamical equation

$$\lim_{x'\to x} K_x W_{2m+2}(x'xx_1\ldots x_{2m})+4\lambda_0 W_{2m+4}(xxxxx_1\ldots x_{2m})=0 \qquad (2.12)$$

displayed diagrammatically in figure 2.

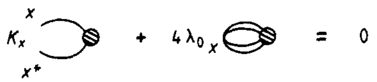

Figure 2. The diagrammatic description of the additional equation (2.12).

We now have a situation in which, once $W_2^{(0)}$ is given, all other $W_{2n}^{(p)}$ cease to be arbitrary†.

There is no analogue of (2.12) for the IVM, so we must assume that, in this case, the IVM is untypical. A discussion of (2.12) is the essential content of the next section.

3. The augmented theory

We need an alternative approach to equations (2.9). One possibility proposed by Klauder for the scale-invariant case of $\beta = 1$ only, is to replace $Z'[h]$ by the augmented generating functional

$$Z'[h,j]=\int \mathscr{D}[\phi]\mathscr{D}[\chi]\exp-\int dx\,(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\lambda_0\phi^4+\tfrac{1}{2}\eta\phi^2\chi^2-ih\phi-ij\chi). \qquad (3.1)$$

We see that, on setting $j=0$ in (3.1) and performing the Gaussian integration, we reproduce $Z'[h]$ of (2.1).

However, the equations of motion satisfied by (3.1) are expected (by virtue of the translation covariance) to be

$$\left\{h(x)+K_x\frac{\delta}{\delta h(x)}-\eta\frac{\delta^3}{\delta h(x)\delta j(x)^2}-4\lambda_0\frac{\delta^3}{\delta h(x)^3}\right\}Z'[h,j]=0 \qquad (3.2)$$

and

$$\left\{j(x)-\eta\frac{\delta^3}{\delta h(x)^2\delta j(x)}\right\}Z'[h,j]=0. \qquad (3.3)$$

In terms of the unconnected Green functions

$$G_{m,n}(x_1\ldots x_m;y_1\ldots y_n)=\frac{1}{i^{m+n}}\left.\frac{\delta^{m+n}Z'[h,j]}{\delta h(x_1)\ldots\delta h(x_m)\delta j(y_1)\ldots\delta j(y_n)}\right|_{h=j=0} \qquad (3.4)$$

the equations following from (3.2) and (3.3) are

$$\sum_{r=1}^{q+1}\delta(x-y_r)G_{p,q-1}(x_1\ldots x_p;y_1\ldots\hat{y}_r\ldots y_q)-\eta G_{p+2,q+1}(xxx_1\ldots x_p;xy_1\ldots y_q)$$

$$=0 \qquad \text{even } p\geqslant 0,\ \text{odd } q\geqslant 1 \qquad (3.5)$$

† We have not been concerned here with the 'ascending' problem of how to determine the W_{2n}, $n>1$, in terms of W_2. Nouri-Moghadam and Yoshimura (1978a) comment on this problem.

and

$$\sum_{r=1}^{m} \delta(x-x_r)G_{p-1,q}(x_1\ldots\hat{x}_r\ldots x_p;\, y_1\ldots y_q)+K_xG_{p+1,q}(xx_1\ldots x_p;\, y_1\ldots y_q)$$
$$-\eta_0G_{p+1,q+2}(xx_1\ldots x_p;\, xxy_1\ldots y_q)-4\lambda_0G_{p+3,q}(xxxx_1\ldots x_p;\, y_1\ldots y_q)$$
$$=0 \qquad \text{odd } p\geqslant 1,\ \text{even } q\geqslant 0. \tag{3.6}$$

Equations (3.5) and (3.6), displayed in figure 3, contain equations (2.9) as a special case. Also, from (3.5) ($p=0, q=1$) and (3.6) ($p=1, q=0$) we have

$$K_xG_2(xy)+\eta[G_2(xx;\,xy)-G_2(xy;\,xx)]+4\lambda_0G_4(xxxy)=0 \tag{3.7}$$

implying the critical equation (2.12) on taking $x=y$†.

In understanding how equations (3.7) (and (2.12)) inevitably arise in the augmented formalism it is sufficient to consider the *pseudo-free* theory ($\lambda_0=0$) for which equation

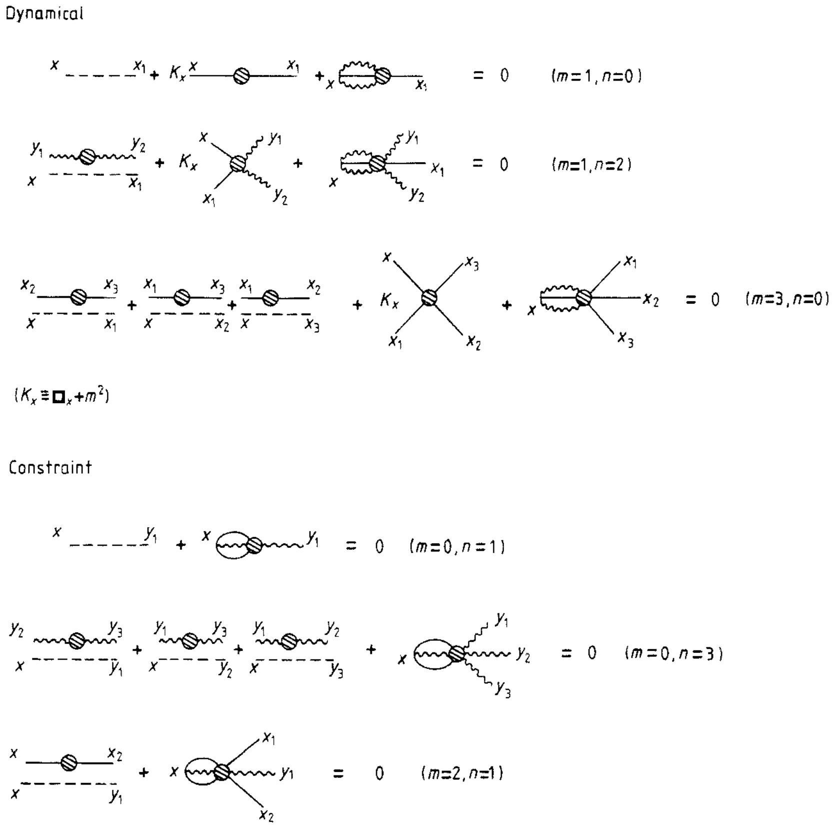

Figure 3. The diagrammatic description of the augmented equations (3.5) and (3.6). The straight lines correspond to ϕ fields, the wavy lines to the auxiliary χ fields. Circles now denote unconnected Green functions.

† For some reason it was argued in Nouri-Moghadam and Yoshimura (1978a) that the augmented formalism contained no new information.

(2.12) becomes

$$\lim_{x\to y} K_x G_2(x-y) = 0. \tag{3.8}$$

Until now this has been considered an unacceptable constraint. In paper III of this series we examine the pseudo-free equation (3.8) in detail. We summarise our conclusions below.

Firstly, by assuming the spectral representation for G_2

$$G_2(x-y) = \int đk \exp[ik(x-y)] \int \frac{d\sigma\, \rho(\sigma)}{k^2+\sigma} \tag{3.9}$$

equation (3.8) can be formally expressed as

$$m_0^2 = \bar{m}^2 - \delta(0)/G_2(0) \tag{3.10}$$

where

$$\bar{m}^2 = \frac{1}{G_2(0)} \int d\sigma\, \rho(\sigma)\sigma G_0(0; m^2) \tag{3.11}$$

with

$$G_0(x; m^2) = \int \frac{đk\, e^{ik\cdot x}}{k^2+m^2}. \tag{3.12}$$

Equation (3.8), via (3.10), can be understood as a self-consistent additive mass renormalisation. This interpretation is given more credence by the following arguments, each of which is given more fully in III.

3.1. *Diagrammatic expansion of the pseudo-free augmented generating functional*

Performing the Gaussian ϕ integration in the *pseudo-free* augmented generating functional

$$Z[h,j] = \int \mathscr{D}[\phi]\mathscr{D}[\chi] \exp - \int dx \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 + \tfrac{1}{2}\eta\phi^2\chi^2 - ih\phi - ij\chi\right) \tag{3.12a}$$

gives

$$Z[h,j] = \int \mathscr{D}[\chi] \exp - \left(\mathfrak{A}[\chi,h] - i\int j\chi\right). \tag{3.13}$$

The effective action $\mathfrak{A}$ for the χ field is a non-polynomial interaction with non-local vertices that describes the hard-core effect due to the scale-invariant measure.

The generating functional $W[h]$ for the connected ϕ Green functions is seen, from (3.13), to be the sum of vacuum diagrams constructed from the action $\mathfrak{A}$ (with h-dependent vertices).

We have already seen in (3.10) that the additional constraint (3.8) has implications for mass renormalisation. On analysing the two-point function, we can establish a hierarchy in the singularity structure, the most singular contributions being point mass insertions, giving a renormalised mass of the form

$$m^2 = m_0^2 + a\frac{\delta(0)}{G_0(0; m_0^2)} + b\frac{L_2}{G_0(0; m_0^2)}\left(\frac{\delta(0)}{G_0(0; m_0^2)}\right)^2 + c\left(\frac{L_2}{G_0(0; m_0^2)}\right)^2\left(\frac{\delta(0)}{G_0(0; m^2)}\right)^3$$
$$+ \ldots + d\frac{L_3}{G_0(0; m_0^2)}\left(\frac{\delta(0)}{G_0(0; m_0^2)}\right)^3 + \ldots + e\frac{L_4}{G_0(0; m_0^2)}\left(\frac{\delta(0)}{G_0(0; m_0^2)}\right)^4 + \ldots \tag{3.14}$$

where L_n is the n-link loop constricted from the propagators $G_0(x; m_0^2)$.

It is straightforward to see that this series is compatible with the self-consistent mass renormalisation

$$m^2 = m_0^2 + \delta(0)/G_0(0; m^2) \tag{3.15}$$

that follows from (3.10) on retaining only the most singular pole term and neglecting the continuum i.e. on making the approximation

$$\rho(\sigma) = \delta(\sigma - m^2). \tag{3.16}$$

We have thus identified the mass renormalisation implied by (3.8) as arising naturally from the hard-core interaction associated with the change of measure.

Furthermore, we have identified the *subtraction* procedure of (2.8) with *normal ordering* of χ in (3.13), i.e. with the preservation of singular contributions in the non-polynomial hard-core interaction. From our knowledge of non-polynomial interactions (and for other reasons) we are unwilling to invoke this.

3.2. *Functional differential equations for the augmented theory*

Consider the translation-covariant branching equations (3.5) and (3.6) for the pseudo-free theory ($\lambda_0 = 0$). They are represented diagrammatically in figure 3. We see that the constraint equations (3.5) imply a high degree of factorisation. For example,

$$G_{p,0}(x_1 \ldots x_p)G_{2,2}(xx; xy) = G_{p+2,2}(xxx_1 \ldots x_p; xy). \tag{3.17}$$

Furthermore, the constraint

$$\delta(x-y) - \eta G_{2,2}(xx; xy) = 0 \tag{3.18}$$

suggests that, at the level of leading singularities,

$$G_{2,2}(xx; xy) \approx G_{2,0}(xx)G_{0,2}(xy) \tag{3.19}$$

whence

$$G_{0,2}(xy) \propto \delta(x-y). \tag{3.20}$$

This, in turn, implies from (3.17)

$$G_{p+2,2}(xxx_1 \ldots x_p; xy) \propto G_{p,0}(x_1 \ldots x_p)\delta(x-y) \tag{3.21}$$

(at the level of leading singularities).

These factorisations, and more, can be achieved by assuming that, again at the level of leading singularities, $Z'[h, j]$ factorises as

$$Z_0'[h, j] \approx H[h]J[j]. \tag{3.22}$$

Inserting this in the *pseudo-free* version ($\lambda_0 = 0$) of (3.2) and (3.3) gives

$$\left[h(x) - \left(K_x + \frac{\delta(0)}{G_2(0)}\right)\frac{\delta}{\mathrm{i}\delta h(x)}\right]H[h] \approx 0. \tag{3.23}$$

That is, at the level of leading singularities, we have an effective 'free' theory (in the sense of no continuum contributions) with mass m satisfying

$$m^2 = m_0^2 + \frac{\delta(0)}{G_2(0)} = m_0^2 + \frac{\delta(0)}{G_0(0;\, m^2)}. \tag{3.24}$$

Thus, again by retaining the effects of leading singularities, we have reproduced (3.16).

3.3. *Large-N limit of the O(N)-invariant pseudo-free theory*

The analysis so far has isolated the most singular contributions (to the diagrammatic expansion, for example) as driving the underlying physics. This is a common occurrence in $1/N$ expansions, suggesting that we consider the O(N)-invariant generalisation of the single-scalar pseudo-free theory.

That is, we generalise the generating function (3.12) to (imposing $j = 0$)

$$Z'[\boldsymbol{h}] = \int \mathscr{D}[\boldsymbol{\phi}]\mathscr{D}[\boldsymbol{\chi}] \exp - \int \mathrm{d}x[\tfrac{1}{2}(\nabla\boldsymbol{\phi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\phi}^2 + \tfrac{1}{2}(\eta/N)\boldsymbol{\phi}^2\boldsymbol{\chi}^2 - \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\phi}] \tag{3.25}$$

by introducing N auxiliary fields $\boldsymbol{\chi}$. On taking the large-N limit of (3.25) in the usual way we find that the theory is described by an effective potential V satisfying

$$V[\boldsymbol{\phi}^2, m^2, \sigma]/N = \tfrac{1}{2}m^2\boldsymbol{\phi}^2/N - \tfrac{1}{2}\sigma(m^2 - m_0^2) + \tfrac{1}{2}\int \mathrm{d}k \ln \sigma(k^2 + m^2) \tag{3.26}$$

where m^2 and σ are auxiliary *fields*. At the minimum of V, m becomes the common mass of the ϕ fields, satisfying

$$m^2 = m_0^2 + \delta(0)/G_0(0;\, m^2). \tag{3.27}$$

That is, the *approximate* results (3.15) and (3.23) that follow from (3.16) become *exact* in the large-N limit. Yet again, the large-N limit provides a natural way to organise diagrams according to their degree of singularity.

However, despite the importance of equation (3.27) we have yet to interpret it. Suppose the pseudo-free theory is in d space–time dimensions. Treated as the limit of a $\lambda|\phi|^n$ theory as $\lambda \to 0$, the failure of the canonical theory forces us to adopt the scale-covariant theory for $d > 2n/(n-2)$, but *a priori* it is defined for all d.

If, as an intermediate step, we regularise $\delta(0)$ and $G_0(0;\, m^2)$ with the momentum cut-off $|k| < \Lambda$ we see that (3.27) can be re-expressed in terms of finite quantities in the $\Lambda \to \infty$ limit whenever $d \geq 4$. For example, for $d > 4$ we have

$$\delta(0)_\Lambda/G_0(0;\, m^2)_\Lambda = a_d\Lambda^2 + b_d m^2 \tag{3.28}$$

with a_d, b_d *finite*, Taking $\mu^2 = (m_0^2 + a_d\Lambda^2)(1 - b_d)^{-1}$ finite as $\Lambda \to \infty$ then makes (3.27) well defined, as $m^2 = \mu^2$. On the other hand, for $d < 4$ it is *not* possible to make (3.27) well defined.

We find this first dimension-specific result encouraging since it is in $d \geq 4$ dimensions that we are most interested. More details are given in III.

We conclude this section with the reminder that $\beta = 1$ (i.e. a scale-invariant measure) was crucial in suggesting the dynamical importance of (3.8) as a necessary additional piece of information (and enabling it to have a diagrammatic expansion via auxiliary fields).

However, once we have used this equation to motivate the organisation of ultraviolet singularities as in the approximations of §§ 3.2 and 3.3 we shall see that it is straightforward to relax the condition $\beta = 1$ in the context of such orderings. Thus, for example, (3.27) would be replaced by

$$m^2 = m_0^2 + \beta\delta(0)/G_0(0, m^2).$$

The conclusions following (3.27) remain unchanged.

4. Stability

We digress to examine one of the global problems of changing the measure. Staying with the Euclidean scalar theory we see that, without introducing auxiliary fields, we can express $Z'[h]$ of (2.1) in a translation-covariant way as†

$$Z'[h] = \int \mathscr{D}[\phi] \exp -\frac{1}{\hbar} \int \mathrm{d}x \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 + \tfrac{1}{2}\beta\hbar\delta(0)\ln\phi^2 - h\phi\right). \qquad (4.1)$$

That is, the potential has acquired the additional term $\frac{1}{2}\beta\hbar\delta(0)\ln\phi^2$. Although not present in the classical limit $\hbar \to 0$, for all $\hbar \neq 0$ it is unbounded below at vanishing ϕ, suggesting that we may have problems with the underlying stability of the theory.

Since this unboundedness is independent of the $\lambda_0\phi^4$ term it is sufficient to examine the stability of the *pseudo-free* theory. This is the content of IV, which we summarise below.

The main tool for determining stability is the effective potential. There are two circumstances in which we can perform explicit calculations. The first is the large-N limit of the O(N)-invariant pseudo-free theory that we have already mentioned, and which will become increasingly important in our analysis. It is more illuminating to rewrite Z' for this case as

$$Z'[h] = \int \mathscr{D}[\boldsymbol{\phi}] \exp -\frac{1}{\hbar} \int \mathrm{d}x \left(\tfrac{1}{2}(\nabla\boldsymbol{\phi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\phi}^2 + \tfrac{1}{2}N\beta\hbar\delta(0)\ln(\boldsymbol{\phi}^2/N) - \boldsymbol{h}\cdot\boldsymbol{\phi}\right). \qquad (4.2)$$

In the large-N limit we recover the effective potential V of (3.26) (except that the last term is multiplied by $\hbar$).

The contribution to V that is of concern is the term

$$V_0 = \tfrac{1}{2}N\beta\hbar\delta(0)\ln\sigma \qquad (4.3)$$

where σ is formally related to $\boldsymbol{\phi}^2$ and m^2 by

$$\sigma = \frac{\boldsymbol{\phi}^2}{N} + \hbar\int\frac{\mathrm{d}k}{k^2 + m^2}. \qquad (4.4)$$

If we were to set $\hbar$ to zero in (4.4) we would reproduce the $N\hbar\ln\boldsymbol{\phi}^2$ term that would be a signal for instability. However, for $\hbar \neq 0$, σ diverges positively and is driven

† We differ from (2.1) in restoring $\hbar$ (set to unity previously) and replacing $\mathrm{i}h\phi$ by $h\phi$, for reasons that will become clear later.

away from zero. Whether the instability is avoided or not depends on our ability to renormalise the effective potential V. (So far we have only examined the renormalisation of the equation (3.27) for the minimum of V.)

For $d \geqslant 4$ dimensions it is possible to renormalise V. To see this we note that the auxiliary field m^2 in (4.4) is constrained to satisfy

$$m^2 = m_0^2 + \beta\hbar\delta(0)/\sigma \tag{4.5}$$

$$= m_0^2 + \frac{\beta\delta(0)}{G_0(0;\, m^2) + \boldsymbol{\phi}^2/\hbar N}$$

$$= m_0^2 + \frac{\beta\delta(0)}{G_0(0;\, m^2)}\left(1 - \frac{\boldsymbol{\phi}^2}{\hbar N G_0(0;\, m^2)} + \mathrm{O}(G_0^{-2})\right). \tag{4.6}$$

Regularising with a momentum cut-off $|k| < \Lambda$ as before in d dimensions

$$\begin{aligned} &\delta(0)_\Lambda/G_0(0,\, m^2)_\Lambda = \mathrm{O}(\Lambda^2)\\ &\delta(0)_\Lambda/G_0^2(0,\, m^2)_\Lambda = \mathrm{O}(\Lambda^{4-d})\\ &\delta(0)_\Lambda/G_0^3(0,\, m^2)_\Lambda = \mathrm{O}(\Lambda^{6-2d}) \end{aligned} \tag{4.7}$$

etc.

Thus, for $d > 4$ dimensions m^2 becomes $\boldsymbol{\phi}^2$ independent as $\Lambda \to \infty$, as do the second and third terms in (3.26). This gives

$$V(\boldsymbol{\phi}^2) = \tfrac{1}{2}m^2\boldsymbol{\phi}^2 \tag{4.8}$$

with m^2 satisfying (3.27). That is, the large-N effective potential for the pseudo-free theory is that of a *free* theory of mass m. The instability is genuinely avoided. For $d < 4$ dimensions it is not possible to renormalise†. This dimension dependence is again encouraging.

Although the large-N limit (as the precursor to a $1/N$ expansion) is potentially the most productive approach, we should not neglect any avenue at this stage. The second situation about which we have some knowledge is the 'strong-coupling'‡ limit, in which we expand about the IVM (see paper II) for which $\beta = 1$. If we were to adopt the over generous analytical regularisation scheme of Kovesi-Domokos (1976) we can calculate the effective potential of the strong-coupling theory exactly, since it corresponds to a tree theory. At the least, this may be an important part of the whole picture. Again, instability is avoided by virtue of the operator product expansion implied by the scale-covariant formalism. For constant source j (which is all that is needed to define the effective potential) the generating functional $W[j]$ for *connected* Green functions is itself of exponential form

$$W[j] = \alpha\hbar b\int_0^\infty \frac{\mathrm{d}u}{u}[\cosh(uj/\hbar) - 1]\exp(-b\bar{m}^2u^2/2\hbar) \tag{4.9}$$

where $\bar{m}^2$ is related to m_0^2 by $b\bar{m}^2 = \delta(0)m_0^2$ with b an arbitrary mass scale.

From (4.9) it follows that $\phi = \delta W/\delta j$ is monotonic and that the effective potential $V(\phi)$ is bounded below. That is, again we have no sign of instability.

† For $d = 4$ we also reproduce equation (4.8).

‡ In this case, the large m_0^2 limit.

Furthermore, we note that the pseudo-free theory (with the regularisation of Kovesi-Domokos (1976)) reduces to the *free* theory of a scalar field on taking the limit $b\hbar \to \infty$. This can be understood as either corresponding to working at energies very much lower than the mass scale b, or as moving away from the classical limit $\hbar = 0$. i.e. the high-temperature expansion.

So far we have only discussed the Euclidean theory, arguing that stability is related to the boundedness of the Euclidean effective potential. This is not strictly true, since the effective potential only has the interpretation of an energy density for the Minkowski theory (Coleman 1975).

In the large-N limit nothing changes on going from the Euclidean to the Minkowski theory. The effective potential remains real and the potentially dangerous origin of σ space is avoided by virtue of the 'quantum corrections'.

The situation is not so clear in the 'strong-coupling' limit, for which there is more than one way of continuing. If we accept the regularisation scheme of Kovesi-Domokos (1976) exactly, one possibility that arises is that the effective potential V does indeed possess a branch with the $\ln \phi^2$ singularity that we wish to avoid. However, the mechanism for avoiding instability exists in principle in that the effective potential displays another branch (bounded below) akin to the branch of the Euclidean theory. Only if the regularisation of Kovesi-Domokos (1976) is exact is tunnelling inevitable. Because of the lack of any problem in the large-N case, we assume that there is no real problem here, although we have no knowledge of how to include terms absent in Kovesi-Domokos (1976) that could, for example, give rise to a phase transition.

5. The interacting scale-covariant theory

So far our analysis has essentially been restricted to the pseudo-free theory. In general we do not know how to incorporate a $\lambda\phi^4$ self-interaction. The pseudo-perturbation expansion requires a greater knowledge of the pseudo-free theory than we possess.

However, the large-N limit discussed in the previous sections suggests a way forward since we know that, for the *canonical* $\lambda(\boldsymbol{\phi}^2)^2$ theory, it resums the most singular self-interaction diagrams (Coleman *et al* 1974). The large-N limit (as a first step in a $1/N$ expansion) for the interacting scale-covariant theory can therefore be expected to pit the most singular terms of the self-interaction against the most singular terms of the 'hard-core', in an *additive* way. With luck, the weaker can be subsumed in the stronger using conventional $1/N$ renormalisation techniques.

As a first step, we wish to determine the equations for the ground-state of the large-N limit of the O(N)-invariant theory with

$$Z'[\boldsymbol{h}] = \int \mathscr{D}[\boldsymbol{\phi}] \exp -\frac{1}{\hbar} \int \mathrm{d}x \left[\tfrac{1}{2}(\nabla\boldsymbol{\phi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\phi}^2 + (\lambda_0/N)(\boldsymbol{\phi}^2)^2 + \tfrac{1}{2}\hbar\beta N\delta(0)\ln(\boldsymbol{\phi}^2/N) + \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\phi}\right]. \tag{5.1}$$

Since (5.1) is now translation covariant it is permissible to introduce yet a further auxiliary field to replace $(\boldsymbol{\phi}^2)^2$ by a Yukawa interaction in the standard way†, whereupon the $\boldsymbol{\phi}$ integration can be performed.

† It is easy to see from Nouri-Moghadam and Yoshimura (1978b) that the scale-covariant $\lambda\phi^4$ and Yukawa theories are inequivalent.

On taking the large-N limit the self-consistent mass renormalisation equation now becomes†

$$m^2 = m_0^2 + 4\lambda_0 \hbar G_0(0, m^2) + \beta\delta(0)/G_0(0, m^2). \tag{5.2}$$

The first two terms in this tree theory describe the Hartree-like mass gap of the canonical theory, to which is *added* the hard-core contribution. This addition is the remarkable property of the $1/N$ expansion that we had anticipated.

We have already seen that our ability to make sense of the pseudo-free equation (3.27) was dimension specific. It will be even more so for (5.2).

On imposing the momentum cut-off $|k| < \Lambda$, we see that (neglecting logarithms)

$$m^2 = m_0^2 + \lambda_0 \mathrm{O}(\Lambda^{d-2}) + \mathrm{O}(\Lambda^2) \tag{5.3}$$

where d is the number of space–time dimensions. Thus, for the more interesting case of $d > 4$, the hard-core is *less* singular than the self-interaction. If the large-N expansion enables us to control the ultraviolet divergences of the self-interaction, the hard-core may present only minor difficulty.

In general, it is well known that the $1/N$ expansion is less singular than the $\hbar$ expansion. For the case in point it is known (Rembiesa 1978) that the $1/N$ expansion is renormalisable in $d < 6$ dimensions, but not for $d \geq 6$‡.

We thus have three possibilities.

(i) $d \geq 6$ dimensions. In this case the pseudo-free equation (3.27) is renormalisable but the interacting equation (5.2) is not.

(ii) $4 \leq d < 6$ dimensions. In this case both the pseudo-free equation (3.27) and the interacting equation (5.2) are renormalisable.

(iii) $d < 4$ dimensions. In this case the pseudo-free equation (3.27) is not renormalisable but the interacting equation (5.2) may be, the self-interaction regularising the *more* singular hard-core.

We find that the interacting equation (5.2) is renormalisable for $d = 3, 4, 5$ when it can be expressed in the form

$$m^2 = \mu^2 + 4\lambda \hbar G_R(0, m^2) \tag{5.4}$$

with $\mu^2(m_0^2, \lambda_0, \Lambda)$, $\lambda(m_0^2, \lambda_0, \Lambda)$ finite in the $\Lambda \to \infty$ limit and G_R the finite part of G_0. That is, the hard-core effect has been absorbed into the self-interaction. Whereas λ is zero for λ_0 zero in $d = 4, 5$ dimensions it is not defined for zero λ_0 in $d = 3$ dimensions. Details are given in V.

For $\lambda \neq 0$, equation (5.4) is the conventional (canonical) large-N mass gap equation.

Of course, the renormalisation of (5.2) is just a first step. The next step is to renormalise the whole effective potential. The situation is more complicated than for the pseudo-free theory, and details are given in V. We confine ourselves to a few brief comments.

Firstly, the ultraviolet singularities of the 'hard-core' interaction can be absorbed into the self-interaction for $d = 3, 4, 5$ dimensions for the whole effective potential in the large-N limit.

† The equations giving rise to (5.2) cease to be consistent if we drop the kinetic term. This explains the unorthodox explicit N-behaviour demanded of the IVM (Klauder and Narnhoffer 1976). Equation (5.2) is the large-N limit of (2.12).

‡ From Klauder's viewpoint it is incorrect to assume that the canonical theory exists in $d > 4$ space–time dimensions. The $1/N$ expansion in $d > 4$ dimensions can therefore be understood only in the above context.

For $d=4, 5$ dimensions we find that, after this absorption, the large-N effective potential is just that of the 'canonical' theory (Coleman *et al* 1974, Rembiesa 1978). That is, there is no β dependence. On the other hand, for $d=3$ dimensions the effective potential is β dependent, despite the β independence of the 'mass-gap' equation. This allows the possibility of a β-dependent phase structure.

Yet again we have a dimension-specific result that indicates that the non-canonical scale-covariant quantisation is unambiguously possible only when it is obligatory†. However, we accept that there are problems with the large-N limit for $d=4$ dimensions (Linde 1976).

Secondly, we can extend the results to incorporate $\lambda(\boldsymbol{\phi}^2)^p$ theories. In general we find that the hard-core ultraviolet singularities are *less* singular than those of the self-interaction (in the large-N limit) only when the $\hbar$ expansion would be *non*-renormalisable.

Finally, even this success is just the first step in constructing a fully renormalised series in $1/N$. We accept that the canonical $1/N$ expansions can be problematical (Linde 1976). Insofar as the non-canonical scale-covariant series differs from the canonical series such difficulties are potentially avoidable.

However, it is possible that for $\lambda_0 \neq 0$ both canonical (translation-covariant) and non-canonical (scale-invariant) quantisations give rise to *identical* $1/N$ series for $d>4$ dimensions, despite the very different structure of the branching equations. This could happen if the heuristic identity $(\lambda_0 \neq 0)$

$$\int \mathscr{D}[\phi] \exp\left(-\frac{1}{\hbar}A[\phi]\right) = \int \mathscr{D}'[\phi] \exp\left(-\frac{1}{\hbar}A[\phi]\right) \tag{5.5}$$

that we used to motivate the ideas of discontinuous perturbations (Klauder 1979b) is respected by the $1/N$ expansion‡.

This possibility is currently under investigation. Nonetheless, we are beginning to make quantitative analytic progress for a class of theories that, until now, have been notoriously elusive.

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† We have already observed that scale-covariant quantisation has been invoked as a way to handle non-renormalisable theories (i.e. $d>4$ dimensions for the case above). One possible interpretation of non-renormalisable theories is that they are trivial (Aizenman 1981, Frohlich 1982). By the same token, canonical $\lambda\phi^4$ in $d=4$ dimensions is also trivial. Non-canonical quantisation is therefore obligatory for $d=4$ dimensions if we require a non-trivial result (Klauder 1981a, b).

‡ We have already observed that it is *not* respected by strong-coupling expansions that are based on the IVM.

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